J. Fluid Mech. (1976), *vol.* 78, *part* 2, *pp.* 209–216
Printed in Great Britain

A conservation law for internal gravity waves

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(Received 24 October 1975 and in revised form 20 July 1976)

The scaled vorticity $\mathbf{\Omega}/N$ and strain $\nabla\zeta$ associated with internal waves in a weak density gradient of arbitrary depth dependence together comprise a quantity that is conserved in the usual linearized approximation. This quantity I is the volume integral of the dimensionless density $D_I = \frac{1}{2}[\Omega^2/N^2 + (\nabla\zeta)^2]$. For progressive waves the 'kinetic' and 'potential' parts are equal, and in the short-wavelength limit the density D_I and flux $\mathbf{F}_I$ are related by the ordinary group velocity: $\mathbf{F}_I = D_I \mathbf{c}_g$. The properties of D_I suggest that it may be a useful measure of local internal-wave saturation.

1. Introduction

A dynamical quantity is a local invariant if its density D and associated flux $\mathbf{F}$ obey a conservation law

$$\partial D/\partial t + \nabla . \mathbf{F} = 0. \tag{1.1}$$

Energy is a well-known invariant in the absence of viscosity, and for small-amplitude internal gravity waves its density and flux are, in terms of the fluid density ρ, velocity $\mathbf{u}$ and vertical displacement ζ,

$$D_E = \tfrac{1}{2}\rho(u^2 + N^2\zeta^2), \quad \mathbf{F}_E = p'\mathbf{u}, \qquad (1.2), (1.3)$$

where p' is the fluctuating part of the pressure and $N(z)$ is the Brunt–Väisälä frequency.

The purpose of this communication is to identify another invariant I for internal waves, associated not with the quantities ζ and $\mathbf{u}$ themselves, but with their spatial gradients. The density of this quantity is dimensionless, and is composed of a kinetic part involving the wave vorticity $\mathbf{\Omega} = \nabla \times \mathbf{u}$ and a kinematic, 'potential' part involving the strain vector $\nabla\zeta$:

$$D_I = \tfrac{1}{2}[\Omega^2/N^2 + (\nabla\zeta)^2]. \tag{1.4}$$

Under the Boussinesq approximation, and in the limit of small amplitude, this quantity will be shown to obey the conservation law

$$\partial D_I/\partial t + \nabla . \mathbf{F}_I = 0, \tag{1.5}$$

where the associated flux is defined by

$$\mathbf{F}_I = -\zeta \partial \mathbf{u}/\partial z. \tag{1.6}$$

It is worth emphasizing that the law holds good for arbitrary $N(z)$, subject only to the overall Boussinesq constraint that the total change in inertial density over the fluid column be small.

This conservation law is weaker than that for energy, since it breaks down for general nonlinear flow; however, when the linearized approximation is valid, the behaviour of I is analogous to that of energy in several respects. The kinetic and potential parts of I are equal for progressive waves,

$$\tfrac{1}{2}\int (\Omega^2/N^2)\,d\mathbf{x} = \tfrac{1}{2}\int (\nabla\zeta)^2 d\mathbf{x}, \tag{1.7}$$

and are therefore global invariants individually. In the short-wavelength limit, the local kinetic and potential densities are equal and the flux is equal to the density multiplied by the ordinary dispersive group velocity, i.e.

$$\mathbf{F}_I = D_I\mathbf{c}_g, \tag{1.8}$$

signifying that I propagates locally with the wave group.

2. Derivation

The linearized momentum equation for an incompressible inhomogeneous fluid can be written as

$$\rho_0\,\partial\mathbf{u}/\partial t + \nabla p' + \hat{\mathbf{z}}g\rho' = 0, \tag{2.1}$$

where p' and ρ' are the instantaneous departures of p and ρ from their values at hydrostatic equilibrium, $p_0(z)$ and $\rho_0(z)$. The buoyancy force, in the direction of the unit vertical vector $\hat{\mathbf{z}}$, has the value

$$g\zeta(d\rho_0/dz) \equiv -\rho_0 N^2(z)\zeta \tag{2.2}$$

when the fluid at depth z has been displaced a small distance ζ above its equilibrium level. In the Boussinesq approximation the inertial density of the fluid is assumed constant, so that the curl of (2.1) divided by ρ_0 is

$$\partial\mathbf{\Omega}/\partial t - N^2(z)\hat{\mathbf{z}}\times\nabla\zeta = 0. \tag{2.3}$$

This is the linearized vorticity equation for internal waves: it signifies that a sloping density surface generates horizontal vorticity whose local effect is to rotate the density surface back towards the horizontal.

The scalar product of this equation with $\mathbf{\Omega}$ is, upon division by $N^2(z)$ and reordering of factors,

$$\partial(\tfrac{1}{2}\Omega^2/N^2)/\partial t + (\hat{\mathbf{z}}\times\mathbf{\Omega}).\nabla\zeta = 0. \tag{2.4}$$

With the aid of the vector identity

$$\hat{\mathbf{z}}\times\mathbf{\Omega} = \hat{\mathbf{z}}\times(\nabla\times\mathbf{u}) = \nabla u_z - \partial\mathbf{u}/\partial z$$

we can write the second term above as

$$(\hat{\mathbf{z}}\times\mathbf{\Omega}).\nabla\zeta = \nabla u_z.\nabla\zeta - (\partial\mathbf{u}/\partial z).\nabla\zeta = \tfrac{1}{2}\partial[(\nabla\zeta)^2]/\partial t - \nabla.(\zeta\,\partial\mathbf{u}/\partial z),$$

the last step in virtue of $u_z = \partial\zeta/\partial t$ and $\nabla.(\partial\mathbf{u}/\partial z) = \partial(\nabla.\mathbf{u})/\partial z = 0$. With this substitution, the equation becomes

$$\tfrac{1}{2}\partial[\Omega^2/N^2 + (\nabla\zeta)^2]/\partial t + \nabla.(-\zeta\,\partial\mathbf{u}/\partial z) = 0, \tag{2.5}$$

completing the demonstration of the conservation theorem (1.4)–(1.6). Note that at any rigid boundary the normal component of flux must vanish, so that I is constant in any volume enclosed by rigid boundaries.

3. Relation between the kinetic and potential terms

Individually, the kinetic and potential parts of the invariant,

$$I_\Omega = \tfrac{1}{2}\int (\Omega^2/N^2)\,d\mathbf{x}, \quad I_\zeta = \tfrac{1}{2}\int (\nabla\zeta)^2\,d\mathbf{x}, \tag{3.1}$$

behave analogously to kinetic and potential energy. They are equal (and constant) in a field of progressive waves, whereas in a field of standing waves, the quantity I oscillates between the kinetic and potential forms twice each wave cycle, so that $I_\Omega = I_\zeta$ on average. The local densities $\frac{1}{2}\Omega^2/N^2$ and $\frac{1}{2}(\nabla\zeta)^2$ depend on the details of the internal-wave field and are not necessarily equal; they differ by a quantity that vanishes in the short-wavelength limit.

To demonstrate these properties, we shall find it convenient to expand the scaled vorticity field $\mathbf{\Omega}/N$ itself in dynamical eigenfunctions. The complete equations of motion consist of the vorticity equation (2.3) and the kinematic relation

$$\partial\zeta/\partial t + \hat{\mathbf{z}}\,.\,\nabla\times\boldsymbol{\psi} = 0, \tag{3.2}$$

written here in terms of the vector stream function $\boldsymbol{\psi}$, defined by

$$\nabla\,.\,\boldsymbol{\psi} = 0, \quad \nabla\times\boldsymbol{\psi} = -\mathbf{u} \tag{3.3}$$

and related to $\mathbf{\Omega}$ by

$$\nabla^2\boldsymbol{\psi} = \mathbf{\Omega}. \tag{3.4}$$

For a field component having horizontal periodicity $\exp(i\mathbf{k}\,.\,\mathbf{x})$, we then have

$$\nabla^2\boldsymbol{\psi}_k = (\partial^2/\partial z^2 - k^2)\,\boldsymbol{\psi}_k = \mathbf{\Omega}_k, \tag{3.5}$$

$$\nabla^{-2}\mathbf{\Omega}_k = -\int G_k(z,z')\,\mathbf{\Omega}_k(z')\,dz' = \boldsymbol{\psi}_k, \tag{3.6}$$

where G_k is the symmetric Green's function for $\nabla^2 = \partial^2/\partial z^2 - k^2$ satisfying $G_k = 0$ at the upper and lower fluid boundaries. In terms of these components, the equations of motion (2.3) and (3.2) read

$$\partial\mathbf{\Omega}_k/\partial t - iN^2(\hat{\mathbf{z}}\times\mathbf{k})\,\zeta_k = 0, \tag{3.7a}$$

$$\partial\zeta_k/\partial t + i(\hat{\mathbf{z}}\times\mathbf{k})\,.\,\nabla^{-2}\mathbf{\Omega}_k = 0. \tag{3.7b}$$

The first of these equations indicates that $\mathbf{\Omega}_k$ is horizontal and transverse to $\mathbf{k}$. If we put $\mathbf{\Omega}_k = (\hat{\mathbf{z}}\times\mathbf{k}/k)\,\Omega_k$, we have for the magnitude Ω_k

$$\partial^2(\Omega_k/N)/\partial t^2 + k^2\Gamma_k(\Omega_k/N) = 0, \tag{3.8}$$

in which Ω_k has been divided by $N(z)$ to symmetrize the integral operator $\Gamma_k = -N\nabla^{-2}N$, defined by the kernel

$$\Gamma_k(z,z') = N(z)\,G_k(z,z')N(z'). \tag{3.9}$$

The solutions to (3.8) are of the form

$$\Omega_k/N \sim \exp(-i\omega_{\mathbf{k}m}t)f_{\mathbf{k}m}(z),$$

where the $f_{\mathbf{k}m}$ are eigenfunctions of the integral equation

$$\Gamma_k f_{\mathbf{k}m} = c^2 f_{\mathbf{k}m}. \tag{3.10}$$

If we further assume that $N(z) > 0$ everywhere, then the $f_{\mathbf{k}m}$ are orthogonal and complete (Courant & Hilbert 1965, pp. 351–362),

$$\int f_{\mathbf{k}m} f_{\mathbf{k}m'} d\mathbf{x} = \delta_{mm'}; \tag{3.11}$$

for simplicity we shall employ periodic boundary conditions in the horizontal, and define the normalization above over the appropriate rectangular volume. The scaled vorticity associated with an arbitrary field of internal waves can now be expanded as

$$\mathbf{\Omega}/N = \mathrm{Re} \sum_{\mathbf{k}m} \epsilon_{\mathbf{k}m}(\hat{\mathbf{z}} \times \mathbf{k}/k) \exp(i\mathbf{k}.\mathbf{x} - i\omega_{\mathbf{k}m} t) f_{\mathbf{k}m}(z), \tag{3.12}$$

where by convention $\omega = ck$ is positive, so that the phase velocity $\mathbf{c}$ is in the direction $\mathbf{k}$. The corresponding expression for the displacement field is

$$\zeta = \mathrm{Re} \sum_{\mathbf{k}m} \epsilon_{\mathbf{k}m} \exp(i\mathbf{k}.\mathbf{x} - i\omega_{\mathbf{k}m} t) g_{\mathbf{k}m}(z), \tag{3.13}$$

where the functions g are related to the functions f through (3.7):

$$f_{\mathbf{k}m} = -c_{\mathbf{k}m}^{-1} N g_{\mathbf{k}m}, \quad g_{\mathbf{k}m} = c_{\mathbf{k}m}^{-1} \nabla^{-2} N f_{\mathbf{k}m}. \tag{3.14a, b}$$

The displacement eigenfunctions g are recognizable as solutions to the more familiar differential equation

$$(\nabla^2 + c^{-2}N^2) g_{\mathbf{k}m} = 0 \tag{3.15}$$

(Phillips 1966, p. 162). This equation can be obtained directly by the application of ∇^2 to (3.14b) and the elimination of $f_{\mathbf{k}m}$.

We can now evaluate the kinetic and potential densities $\frac{1}{2}\Omega^2/N^2$ and $\frac{1}{2}(\nabla\zeta)^2$ in terms of the complex scalar modal amplitudes $\epsilon_{\mathbf{k}m}$. Let

$$\epsilon_{\mathbf{k}m} = |\epsilon_{\mathbf{k}m}| \exp(i\phi_{\mathbf{k}m}),$$

and abbreviate

$$\theta = \mathbf{k}.\mathbf{x} - \omega_{\mathbf{k}m} t + \phi_{\mathbf{k}m}. \tag{3.16}$$

The squared expansion (3.12) for $\mathbf{\Omega}/N$ then becomes

$$\tfrac{1}{2}\Omega^2/N^2 = \tfrac{1}{2}\Sigma(|\epsilon|f)_{\mathbf{k}m}(|\epsilon|f)_{\mathbf{k}'m'}\hat{\mathbf{k}}.\hat{\mathbf{k}}' \cos\theta \cos\theta', \tag{3.17}$$

where $\hat{\mathbf{k}} = \mathbf{k}/k$. With the aid of the identity

$$(\nabla\zeta)^2 = -\zeta\nabla^2\zeta + \tfrac{1}{2}\nabla^2\zeta^2$$

and (3.14) and (3.15), yielding $\nabla^2\zeta$ in terms of the $f_{\mathbf{k}m}$, we obtain the potential density as

$$\tfrac{1}{2}(\nabla\zeta)^2 = \tfrac{1}{2}\Sigma(|\epsilon|cf)_{\mathbf{k}m}(|\epsilon|c^{-1}f)_{\mathbf{k}'m'} \cos\theta \cos\theta' + \nabla.\{\tfrac{1}{4}\nabla\zeta^2\}. \tag{3.18}$$

Integration of (3.17) and (3.18) over volume causes those terms for which either $\mathbf{k} \neq \pm\mathbf{k}'$ or $m \neq m'$ to vanish, so that

$$I_\Omega = \tfrac{1}{4}\Sigma|\epsilon_{\mathbf{k}m}|^2 - \tfrac{1}{4}\Sigma|\epsilon_{\mathbf{k}m}\epsilon_{-\mathbf{k}m}|\cos\theta'', \tag{3.19a}$$

$$I_\zeta = \tfrac{1}{4}\Sigma|\epsilon_{\mathbf{k}m}|^2 + \tfrac{1}{4}\Sigma|\epsilon_{\mathbf{k}m}\epsilon_{-\mathbf{k}m}|\cos\theta'', \tag{3.19b}$$

$$\theta'' = \phi_{\mathbf{k}m} + \phi_{-\mathbf{k}m} - 2\omega_{\mathbf{k}m} t.$$

Note that the time-dependent parts of I_Ω and I_ζ are equal and opposite, hence the sum is constant, as expected:

$$I = I_\Omega + I_\zeta = \tfrac{1}{2}\Sigma|\epsilon_{\mathbf{k}m}|^2. \tag{3.20}$$

The time-dependent terms above represent oscillations between the kinetic and potential forms of I due to standing-wave components of the field. For purely progressive waves, $I_\Omega = I_\zeta$.

For a random stationary field of internal waves, in which the phases θ can be regarded as mutually uncorrelated, the mean kinetic and potential densities consist only of the terms with $\mathbf{k} = \mathbf{k}'$ in expressions (3.17) and (3.18), so that the mean densities differ by

$$\tfrac{1}{2}\overline{(\nabla\zeta)^2} - \tfrac{1}{2}\overline{\Omega^2/N^2} = \tfrac{1}{4}d^2(\overline{\zeta^2})/dz^2. \tag{3.21}$$

The averages above can be defined over field ensembles, or over time for a particular realization of the field. In a field made up of many vertical wavelengths, it is possible for the scale over which $\overline{\zeta^2}$ varies appreciably to be large compared with the contributing wave scales. In such a case (exemplified by internal waves in the deep ocean thermocline) the mean kinetic and potential densities are approximately equal. In the short-wavelength limit they become identical, as is shown below.

4. Short-wavelength approximation

A wave packet with local three-dimensional wavenumber $\mathbf{k}$ can be defined by

$$\zeta = \mathrm{Re}\,(\zeta_0 e^{i\theta}), \quad \mathbf{\Omega}/N = \mathrm{Re}\,(\boldsymbol{\epsilon}_0 e^{i\theta}), \tag{4.1}$$

with $\theta(\mathbf{x}, t)$ such that

$$\mathbf{k} = \nabla\theta, \quad \omega = -\partial\theta/\partial t. \tag{4.2}$$

When the variation of N, ζ_0, and $\boldsymbol{\epsilon}_0$ over the characteristic scales k^{-1} and ω^{-1} is small, the equations of motion are equivalent to

$$\boldsymbol{\epsilon}_0 = -(\hat{\mathbf{z}} \times \mathbf{k})\,\omega^{-1}N\zeta_0 \tag{4.3}$$

and

$$\omega^2 = N^2(\hat{\mathbf{z}} \times \mathbf{k})^2/k^2, \tag{4.4}$$

the latter being the familiar dispersion relation for internal waves in a slowly varying gradient (Phillips 1966, p. 174). We then have

$$\tfrac{1}{2}\Omega^2/N^2 = \tfrac{1}{2}k^2\zeta_0^2\cos^2\theta, \tag{4.5}$$

and

$$\tfrac{1}{2}(\nabla\zeta)^2 = \tfrac{1}{2}k^2\zeta_0^2\sin^2\theta, \tag{4.6}$$

so that the density D_I is locally uniform:

$$D_I = \tfrac{1}{2}k^2\zeta_0^2. \tag{4.7}$$

Using $\boldsymbol{\psi} = \nabla^{-2}\mathbf{\Omega} = -k^{-2}\mathbf{\Omega}$ and $\mathbf{u} = -\nabla \times \boldsymbol{\psi} = -i\mathbf{k} \times \boldsymbol{\psi}$, we find

$$\mathbf{u} = k^{-2}\mathbf{k} \times (\hat{\mathbf{z}} \times \mathbf{k})\,\omega^{-1}N^2\zeta_0\sin\theta,$$

so that the flux is

$$\mathbf{F}_I = -\zeta\,\partial\mathbf{u}/\partial z = -k^{-2}(\mathbf{k}.\hat{\mathbf{z}})\,\mathbf{k} \times (\hat{\mathbf{z}} \times \mathbf{k})\,\omega^{-1}N^2\zeta_0^2\cos^2\theta. \tag{4.8}$$

The vector $\mathbf{F}_I$ is perpendicular to $\mathbf{k}$, i.e. in the plane of the surfaces θ = constant, and perpendicular to $\mathbf{\Omega}$. Consequently its divergence vanishes to first order in the short-wavelength approximation, as does $\partial D_I/\partial t$.

We can recognize in $\mathbf{F}_I$ a factor equal to the ordinary group velocity defined by the dispersion relation (4.4),

$$\mathbf{c}_g \equiv \nabla_k \omega(\mathbf{k}, \mathbf{x}) = (N^2/2\omega)\nabla_k[(\hat{\mathbf{z}}\times\mathbf{k})^2/k^2]$$
$$= -k^{-4}\omega^{-1}N^2(\mathbf{k}.\hat{\mathbf{z}})\,\mathbf{k}\times(\hat{\mathbf{z}}\times\mathbf{k}), \tag{4.9}$$

so that
$$\mathbf{F}_I = (1+\cos 2\theta)\, D_I\,\mathbf{c}_g. \tag{4.10}$$

Averaged over a half-cycle of local phase, the flux is

$$\mathbf{F}_I = D_I\,\mathbf{c}_g, \tag{4.11}$$

signifying that the quantity I propagates with the group. The conservation law, written as a time derivative of D_I along the group trajectory, is then

$$[\partial/\partial t + \mathbf{c}_g.\nabla]\,D_I = -D_I\nabla.\mathbf{c}_g. \tag{4.12}$$

5. Relation to energy

The normal-mode expansion of the total energy,

$$E = \tfrac{1}{2}\int \rho_0(u^2+N^2\zeta^2)\,d\mathbf{x}, \tag{5.1}$$

from (1.1), is straightforward in the Boussinesq approximation, when $\rho_0(z)$ is replaced by the constant value ρ_0; integration by parts yields

$$u^2 \rightarrow -\mathbf{\Omega}.\boldsymbol{\psi} = -\mathbf{\Omega}\nabla^{-2}\mathbf{\Omega},$$

which, together with the relations (3.14), allows the integral above to be put into the form

$$E = \tfrac{1}{2}\rho_0\Sigma|\epsilon_{\mathbf{k}m}|^2 c_{\mathbf{k}m}^2. \tag{5.2}$$

Comparing this expression with (3.20), we see that the ratio of E to I in a given modal component is $\rho_0 c^2$. Since phase velocity is a diminishing function of k and m, the amount of energy associated with a given amount of scaled vorticity or strain is a rapidly decreasing function of total wavenumber, as one would expect.

In nonlinear flow, the modes are coupled and the amplitudes $\epsilon_{\mathbf{k}m}$ are no longer constant. Energy is conserved nevertheless, so that the sum (5.2) remains constant. This suggests that I is in general not constant under nonlinear energy exchange, and that no simple generalization of the conservation law to nonlinear flow exists.

6. Implications

The apparent regularity in the energy spectra of ocean internal waves is widely regarded as evidence that these waves are 'saturated', i.e. limited in amplitude either by sporadic breaking or by nonlinear energy transfer among the spectral components (Garrett & Munk 1975, 1972). In the analogous equilibrium of surface waves, a specific dimensionless quantity can be identified as defining the saturated state: the surface slope, which for all wavelengths determines the degree of nonlinearity and in terms of which the saturated spectrum has a universal, dimensionless form (Phillips 1966, pp. 109–119).

A quantity that plays the same role for internal waves is not so easy to identify because the governing nonlinearities are less well understood for a random wave field, and because the medium has a troublesome third dimension in which the stratification is inhomogeneous. A possible candidate is the density D_I, which is itself a natural measure of the local flow nonlinearity. Because of the average equality $(\nabla\zeta)^2 \simeq \Omega^2/N^2$, the density $D_I = \frac{1}{2}[\Omega^2/N^2 + (\nabla\zeta)^2]$ is numerically equal both to $(\nabla\zeta)^2$ and to Ω^2/N^2. A value of $D_I \sim 1$ implies that either the isopycnal slope $\partial\zeta/\partial x, y$ or the dilation $\partial\zeta/\partial z$ is appreciable, in other words, that the stratified medium is undergoing significant local distortion. Thus the connexion between D_I and the nonlinear regime of the flow equations is direct; what about the connexion with known regimes of instability? In this respect the relation derived in this paper between strain and scaled vorticity is noteworthy, because the quantity Ω^2/N^2 has been shown to have a direct bearing on internal-wave stability in the limit of long horizontal wavelength. In this limit the slopes are small, the flow nearly parallel and the quantity Ω^2/N^2 becomes equal to the inverse Richardson number:

$$D_I \to Ri^{-1} \equiv N^{-2}[(\partial u_x/\partial z)^2 + (\partial u_y/\partial z)^2], \quad k \to 0. \tag{6.1}$$

In parallel flow Ri^{-1} is the appropriate measure of instability: theoretically, $Ri^{-1} < 4$ assures stability (Miles 1961), and conversely, measured values of Ri^{-1} appear not to exceed 4 in ocean internal waves (Sanford 1975).

The quantity D_I is therefore a plausible generalization of Ri^{-1} as an absolute measure of local wave excitation for a multimodal field. The significance of the linearized conservation theorem is that D_I is well behaved as a function of the vertical co-ordinate regardless of the stratification profile $N(z)$ and represents the partitioning over space of an almost-conserved quantity I. In turn, the spatial partitioning is uniquely related to the partitioning of I (and therefore of energy) among the modes $|\epsilon_{\mathbf{k}m}|^2$.

As pointed out in § 5, the conservation law fails in the presence of nonlinear energy exchange among the modes. This property might be turned to advantage, in that the degree of non-conservation of I could furnish an empirical measure of nonlinear exchange rates. Equation (3.20) can be rewritten in terms of the modal energies $E_{\mathbf{k}m}$ defined implicitly in (5.2) as

$$I = \rho_0^{-1} \Sigma E_{\mathbf{k}m} c_{\mathbf{k}m}^{-2}. \tag{6.2}$$

If we imagine a field, free of external energy sources for simplicity, in which energy is cascading from low to high $\mathbf{k}$ and m, in the direction of decreasing c, we infer from (6.2) that I is increasing. The conservation law must accordingly be modified to read

$$\partial D_I/\partial t + \nabla . \mathbf{F}_I = n - d, \tag{6.3}$$

where n is a positive volume source of I proportional to the energy cascade rate and where d is a volume sink representing the effect of dissipative energy loss at the highest wavenumbers. The number $D_I^{-1}n$ can be used to define a local non-linear interaction rate; since both the mean density and the flux of I are simple combinations of measurable local flow quantities, one could experimentally obtain lower-bound estimates of these rates, in the form $D_I^{-1}(n-d)$.

This work was supported by the Defense Advanced Research Projects Agency (DARPA) and monitored by the Office of Naval Research under Contract N00014-73-C-0105.

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